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Author(s)	Nozaki, Yuta
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An explicit relation between knot groups in lens spaces and those in S^3

Yuta Nozaki

Graduate School of Mathematical Sciences, the University of Tokyo

1 Introduction

We consider the symmetry of knots, precisely, free periods. The details of proofs in this article can be found in author's preprint [12]. A knot K in S^3 is said to *have free period* $p \in \mathbb{Z}_{\geq 1}$ if there exists $f \in \text{Diff}(S^3, K)$ such that f^i has no fixed point for $0 < i < p$ and $f^p = \text{id}_{S^3}$, namely, (S^3, K) admits a free action by $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$. Whether a knot K has free period p or not is interesting problem and studied by many people.

We first review previous researches on the existence of free periods. In [7], Hartley proved that the torus knot $T_{m,n}$ has free period p if and only if $\gcd(mn, p) = 1$. For example, the trefoil $T_{3,2}$ does not admit free involution. The Alexander polynomial of a torus knot was used in his proof.

Let K be a knot such that the outer automorphism group $\text{Out}(G(K))$ of $G(K) = \pi_1(S^3 \setminus K)$ is trivial. For instance, 9_{32} , 9_{33} and 24 more prime knots with 10 crossings (and their mirror images) satisfy this condition (see Kawauchi [9, Appendix F.2] or Kodama-Sakuma [10, Table 3.1]). Then it follows from Conner-Raymond [6, Theorem 3.2] and Burde-Zieschang [3] that K has no free period.

The purpose of this article is to deduce the above facts from a single result. Before stating our results, we review previous researches on the uniqueness of free periods. Sakuma [14], Boileau and Flapan [2] independently proved that for an oriented *prime* knot K , if $f, g \in \text{Diff}(S^3, K)$ have free period p , then f is conjugate to g in the subgroup of $\text{Diff}(S^3, K)$ consisting of diffeomorphisms that preserve the orientations of both S^3 and K . They also showed that the same is true for composite knots under a condition regarding “slopes”. Recently, Manfredi [11] gave an interesting example regarding the uniqueness.

In order to state the main result, we describe free periods in another way. Suppose a knot K has period p . Then we obtain a knot $K' = K/\mathbb{Z}_p$ in the lens space $L(p, q) = S^3/\mathbb{Z}_p$ for some integer q coprime to p . Conversely, if K is a preimage of a knot K' under the

covering map $\pi: S^3 \rightarrow L(p, q)$, then (a generator of) the deck transformation group realizes a free period of p . Therefore, we focus on a knot in a lens space, especially, on the fundamental group of its complement.

Theorem 1 (Theorem 2.6). Let K' be a knot in $L(p, q)$ with the connected preimage $K := \pi^{-1}(K')$. Then the image of $\pi_*: \pi_1(S^3 \setminus K) \hookrightarrow \pi_1(L(p, q) \setminus K')$ coincides with $C^p(\pi_1(L(p, q) \setminus K'))$. In particular, the knot group $\pi_1(S^3 \setminus K)$ is a C^p -group (see Definition 2.1).

As a corollary of this result, the facts mentioned above is deduced.

Corollary 1 (Corollary 3.5). A knot K in S^3 with $\text{Out}(G(K)) = 1$ cannot be represented as the preimage of any knot in any lens space.

Corollary 2 (Corollary 3.7). Let $m, n, p \in \mathbb{Z}_{\geq 2}$ with $\gcd(m, n) = 1$. There exists an integer q and a knot K' in $L(p, q)$ such that $\pi^{-1}(K')$ is ambient isotopic to the torus knot $T_{m,n}$ or its mirror image if and only if $\gcd(mn, p) = 1$.

2 Definitions and main theorem

Definition 2.1. For a group G and $p \in \mathbb{Z}_{\geq 1}$, let $C^p(G)$ denote the subgroup of G generated by the set $\{g^p \mid g \in G\} \cup \{[g, h] \mid g, h \in G\}$, where $[g, h] := ghg^{-1}h^{-1}$. A group G is called a C^p -group if there exists a group G' such that $G \cong C^p(G')$.

Remark 2.2. The subgroup $C^p(G)$ coincides with the kernel of the composite map $G \rightarrow G_{\text{ab}} \rightarrow G_{\text{ab}}/pG_{\text{ab}}$.

Remark 2.3. $C^2(G)$ is denoted by G^2 in [17] and by $S(G)$ in [8]. For a prime p , $C^p(G)$ coincides with the first term of the p -lower central series in [16] and with the first term of the derived p -series in [5].

Let $\pi: \Sigma \rightarrow \Sigma'$ be a p -fold cyclic covering, where Σ is an integral homology 3-sphere, and K' be a knot in Σ' with the connected preimage $K := \pi^{-1}(K')$.

Remark 2.4. K is connected if and only if $[K']$ generates $H_1(\Sigma') \cong \mathbb{Z}_p$. The last isomorphism is confirmed by using the five-term exact sequence for the short exact sequence $1 \rightarrow \pi_1(\Sigma) \rightarrow \pi_1(\Sigma') \rightarrow \mathbb{Z}_p \rightarrow 1$.

Lemma 2.5. For Σ' and K' as above, $H_*(\Sigma' \setminus K') \cong \begin{cases} \mathbb{Z} & \text{if } * = 0, 1, \\ 0 & \text{otherwise} \end{cases}$. The homology class represented by a meridian of K' corresponds to $\pm p \in \mathbb{Z}$.

Theorem 2.6. *The image of $\pi_*: \pi_1(\Sigma \setminus K) \rightarrow \pi_1(\Sigma' \setminus K')$ coincides with $C^p(\pi_1(\Sigma' \setminus K'))$.*

Proof. Set $G := \pi_1(\Sigma \setminus K)$ and $G' := \pi_1(\Sigma' \setminus K')$. The covering map π induces the exact sequence

$$1 \rightarrow G \xrightarrow{\pi_*} G' \xrightarrow{\psi} \mathbb{Z}_p \rightarrow 1.$$

Here, ψ factors through $G'^{\text{ab}}/pG'^{\text{ab}} \cong \mathbb{Z}_p$ (Lemma 2.5), and thus $\text{Im } \pi_* = \text{Ker } \psi = C^p(G')$ (Remark 2.2). \square

3 Corollaries

We start this section with a remark in group theory.

Remark 3.1. For a normal subgroup H of a group G , the restriction map $\text{Inn}(G) \rightarrow \text{Aut}(H)$, $\text{Ad}_g \mapsto \text{Ad}_g|_H$, is induced by definition. Furthermore, $\text{Aut}(G) \rightarrow \text{Aut}(H)$ is defined if H is *characteristic*, that is, $f(H) = H$ for all $f \in \text{Aut}(G)$. However, the restriction map $\text{Inn}(G) \rightarrow \text{Inn}(H)$ is not induced in general.

The following lemma is a refinement of the well-known fact [13, 13.5.8] for a complete group H . (A group G is said to be *complete* if the center $Z(G)$ and the outer automorphism group $\text{Out}(G)$ are trivial.)

Lemma 3.2. *Let G, H be groups such that $H \triangleleft G$ and $\text{Ad}_g|_H \in \text{Inn}(H)$ for any $g \in G$. Then the sequence of groups*

$$1 \rightarrow Z(H) \xrightarrow{\phi} H \times C_G(H) \xrightarrow{\psi} G \rightarrow 1$$

is exact, where $C_G(H)$ is the centralizer of H in G , and $\phi(h) := (h, h^{-1})$, $\psi(h, g) := hg$.

Proof. We only confirm the surjectivity of ψ . Let $g \in G$. Then $\text{Ad}_g|_H = \text{Ad}_h$ for some $h \in H$. For any $h' \in H$, we have

$$[h^{-1}g, h'] = h^{-1}gh'g^{-1}hh'^{-1} = \text{Ad}_{h^{-1}}(\text{Ad}_g(h'))h'^{-1} = 1.$$

Hence, $h^{-1}g \in C_G(H)$ and $\psi(h, h^{-1}g) = g$. \square

The next lemma is a generalization of [8, Theorem 1].

Lemma 3.3. *Let G, H be as in Lemma 3.2 and suppose $C^p(G) = H$, $Z(H) = 1$. Then $C^p(H) = H$.*

Proof. By Lemma 3.2, $\psi: H \times C_G(H) \rightarrow G$ and its restriction

$$\psi|: C^p(H) \times C^p(K) \rightarrow C^p(G) = H, \tag{1}$$

are isomorphisms, where $K := C_G(H)$. Since $\psi(C^p(H) \times \{1\}) = C^p(H)$, we have $C^p(K) \cong H/C^p(H)$, and thus $Z(C^p(K)) = C^p(K)$. On the other hand, taking the center of (1), we have $Z(C^p(H)) \times Z(C^p(K)) \cong Z(H) = 1$ and $Z(C^p(K)) = 1$. Hence, we conclude $C^p(K) = 1$, and thus $\psi|$ is the identity map. \square

The quotient $G/C^p(G)$ plays a key role in our argument. The next lemma follows from Remark 2.2 and the homomorphism theorem for the abelianization $G \twoheadrightarrow G_{\text{ab}}$.

Lemma 3.4. *For a group G whose abelianization is isomorphic to \mathbb{Z} , the quotient group $G/C^p(G)$ is isomorphic to \mathbb{Z}_p .*

Corollary 3.5 ([6, Theorem 3.2], [3]). *A knot K in S^3 with $\text{Out}(G(K)) = 1$ is not represented as the preimage of any knot in any lens space.*

Proof. Since $\text{Out}(G(T_{m,n})) \cong \mathbb{Z}_2$ ([15]), K is not a torus knot, and thus $Z(G(K)) = 1$ ([3]). Hence, $G(K)$ is complete.

Assume that there exists a knot K' in $L(p, q)$ whose preimage is isotopic to K . Then we have $G(K) = C^p(\pi_1(L(p, q) \setminus K'))$ by Theorem 2.6. Since $G(K)$ is complete, by Lemme 3.3, we conclude $C^p G(K) = G(K)$. However, this contradicts Lemma 3.4. \square

In order to prove Corollary 3.7, we quote the next lemma without proof.

Lemma 3.6 (see [12, Lemma 3.5]). *Let $m, n, p \in \mathbb{Z}_{\geq 1}$. If there exists a group G satisfying $C^p(G) \cong \mathbb{Z}_m * \mathbb{Z}_n$, $G/C^p(G) \cong \mathbb{Z}_p$ and $|G_{\text{ab}}| = mnp$, then $\gcd(mn, p) = 1$. (Moreover, $H_*(G)$ is isomorphic to $H_*(\mathbb{Z}_m * \mathbb{Z}_n * \mathbb{Z}_p)$.)*

Corollary 3.7 ([7, Theorem 3.1]). *Let $m, n, p \in \mathbb{Z}_{\geq 2}$ with $\gcd(m, n) = 1$. There exist an integer q and a knot K' in $L(p, q)$ such that $\pi^{-1}(K')$ is isotopic to the torus knot $T_{m,n}$ or its mirror image if and only if $\gcd(mn, p) = 1$.*

Proof. If $\gcd(mn, p) = 1$, then a construction of a desired knot K' was given in [7, Theorem 3.1].

Suppose there exists K' as in the statement. We set $\pi'_1 := \pi_1(L(p, q) \setminus K')$. The covering map $\pi: S^3 \setminus T_{m,n} \rightarrow L(p, q) \setminus K'$ induces the exact sequence

$$1 \rightarrow G(T_{m,n}) = \langle a, b \mid a^m = b^n \rangle \xrightarrow{\pi_*} \pi'_1 \rightarrow \mathbb{Z}_p \rightarrow 1.$$

Since the center $Z(G(T_{m,n})) = \langle a^m \rangle = \mathbb{Z}$ is characteristic in $G(T_{m,n})$, the subgroup $N := \pi_*(\langle a^m \rangle)$ of π'_1 is normal. We deduce the exact sequence

$$1 \rightarrow \langle a, b \mid a^m = 1 = b^n \rangle \xrightarrow{\pi_*} \pi'_1/N \rightarrow \mathbb{Z}_p \rightarrow 1$$

from the third isomorphism theorem. By Theorem 2.6, the group $G := \pi'_1/N$ satisfies

$$C^p(G) = C^p(\pi'_1)/N = G(T_{m,n})/\langle a^m \rangle = \mathbb{Z}_m * \mathbb{Z}_n$$

and $G/C^p(G) \cong \mathbb{Z}_p$. Hence, by Lemma 3.6, it suffices to prove $|G_{\text{ab}}| = mnp$.

The five-term exact sequence for

$$1 \rightarrow N \rightarrow \pi'_1 \rightarrow G \rightarrow 1 \quad (2)$$

is as follows:

$$0 \rightarrow H_2(G) \rightarrow \mathbb{Z}_G \rightarrow \mathbb{Z} \rightarrow H_1(G) \rightarrow 0.$$

An observation on a meridian of K' proves $H_1(G) = \mathbb{Z}_{mnp}$ (see [12, Corollary 3.4] for details). \square

Remark 3.8. The above Hartley's result (Corollary 3.7) was extended by Chbili [4] to torus links. In fact, the torus link $T_{m,n}$ has free period p if and only if there exists an integer q such that $\gcd(p, q) = 1$ and $p \mid m - nq$. Note that $\gcd(mn, p) = 1$ implies that the existence of such a q , however, the converse is not true without the assumption $\gcd(m, n) = 1$.

4 Symmetric groups and braid groups

In this section, we suppose $n \geq 3$ and $p \geq 2$ for simplicity. The next lemma follows from Lemma 3.3 and the fact that \mathfrak{S}_n is complete for $n \neq 2, 6$. Note that the case $n = 6$ requires an additional argument (see [12, Appendix A.2]).

Lemma 4.1. *The n th symmetric group \mathfrak{S}_n is a C^p -group if and only if p is odd.*

Even if a group G is a C^p -group, G/H is not necessarily a C^p -group. However, the following lemma asserts that G/H is a C^p -group for a characteristic subgroup H .

Lemma 4.2 (see [17, Theorem 1]). *Let G be a C^p -group and $f: G \rightarrow H$ be a surjective homomorphism whose kernel is a characteristic subgroup of G . Then H is also a C^p -group.*

Proof. Suppose $C^p(G') = G$. Then we have

$$C^p(G'/\text{Ker } f) = C^p(G')/(\text{Ker } f \cap C^p(G')) = G/\text{Ker } f \cong H.$$

Hence, H is a C^p -group. \square

Corollary 4.3. *The n th braid group B_n is not a C^p -group for even p .*

Proof. Since the n th pure braid group $P_n := \text{Ker}(B_n \rightarrow \mathfrak{S}_n)$ is characteristic ([1, Theorem 3]), by Lemma 4.2, it suffices to prove that \mathfrak{S}_n is not a C^p -group for even p . Lemma 4.1 completes the proof. \square

Remark 4.4. One of the definition of B_n is the fundamental group of X_n/\mathfrak{S}_n , where X_n is the configuration space $\{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \ (i \neq j)\}$ of distinct n points in \mathbb{C} , and \mathfrak{S}_n acts on X_n by permutation of coordinates. Therefore, B_n is a \mathbb{C}^p -group if there exists a topological space Y admitting a p -fold cyclic covering $X_n/\mathfrak{S}_n \rightarrow Y$ and satisfying $H_1(X)/pH_1(X) = \mathbb{Z}_p$.

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Graduate School of Mathematical Sciences

The University of Tokyo

Tokyo 153-8914

JAPAN

E-mail address: nozaki@ms.u-tokyo.ac.jp

東京大学大学院数理科学研究科 野崎 雄太